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A note on the total reflexion or transmission of surface waves in the presence of parallel obstacles

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It is shown how a two-dimensional surface wave can be either totally reflected or totally transmitted in the presence of two parallel vertical barriers each containing a small gap. Total transmission of a surface wave past obstacles has been known to occur in many situations in water-wave theory, but total reflexion is a comparatively new phenomenon which could be of practical use in the design of breakwaters.

It would seem reasonable to suppose that, when a two-dimensional plane surface wave is incident upon a finite number of fixed obstacles, then provided that the regions of fluid on either side of the obstacles are connected, part of the wave energy will be reflected and part transmitted in the form of a surface wave of reduced amplitude travelling away from the obstacles. It is shown here that this statement is not true in general and that it is possible to eliminate completely the transmitted wave using as obstacles the simple configuration of two vertical barriers with small symmetric gaps. This total reflexion of the incident wave can be interpreted as an interference effect between the fluid motions just outside and between the barriers.

This phenomenon was first shown to exist in a paper by Evans & Morris (1972), hereafter denoted by I. In that paper the problem of the transmission of surface waves past two vertical parallel barriers immersed to a finite depth below the free surface was considered. This problem was solved using complementary approximations to the solutions of certain integral equations. Because the approximations provided upper and lower bounds to expressions related to the reflexion and transmission coefficients the authors were able to prove that total reflexion was possible for certain configurations of barrier spacing, depth of immersion and incident wavelength. When the barriers were not close together total reflexion occurred at values of the incident wavelength for which the amount of transmitted wave energy was negligible. When the barriers were closely spaced the complementary approximations were poor and only rough bounds could be obtained for the values of the incident wavelength at which total reflexion occurred. These bounds have since been confirmed by Newman (1974), who has considered the case of closely spaced barriers using a matching scheme first employed by Tuck (1971) in a similar context.

The possibility of total *transmission* of an incident wave by obstacles has been

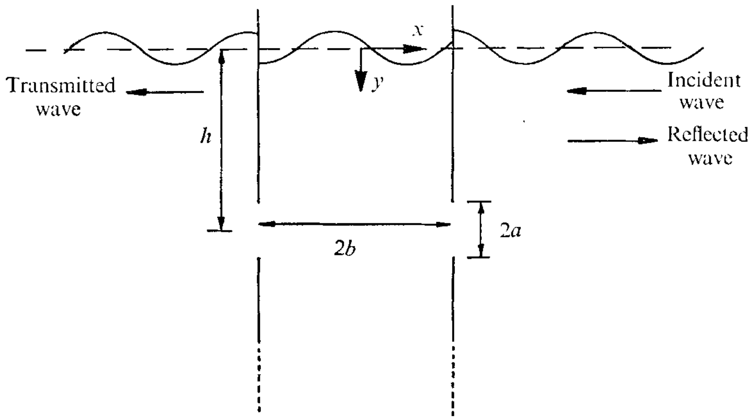

FIGURE 1

known for some time (Newman 1965) and this also occurs in I and in the problem to be considered here.

The problem will be solved using an approximate method valid for small gaps which was used by Packham & Williams (1972) to solve the same problem for a single barrier with a gap. (The same idea has more recently been employed by Leppington & Levine (1973) in considering the transmission of sound waves through a plane screen containing small circular or elliptical holes.) It is possible to obtain the same results using Tuck's matching scheme, which has the advantage of being applicable to more general situations, although the mathematical justification is more difficult than for the method used here.

It is shown that the condition for zero transmission reduces to a single transcendental equation involving parameters based on the incident wavelength, barrier separation and gap depth. It is further shown that, for a given barrier separation and gap depth, there exists an infinity of values of the incident wavelength for which the equation is satisfied, and total reflexion occurs.

The formulation of the problem follows closely that used in I and only an outline is given here. The undisturbed free surface is $y = 0$ and y is measured vertically downwards. The barriers occupy $x = \pm b$, $y > 0$ except for a gap $|y-h| < a$ in each barrier. The usual linearized equations for irrotational incompressible flow give rise to a velocity potential $\Phi(x,y,t) = \mathrm{Re}\{\phi(x,y)\, e^{-i\omega t}\}$, where ω is the incident wave frequency. Then $\phi(x,y)$ is harmonic in $y > 0$ and satisfies

$$K\phi + \partial\phi/\partial y = 0, \quad y = 0, \quad K = \omega^2/g = 2\pi/\lambda,$$
$$\partial\phi/\partial x = 0, \quad x = \pm b, \quad |y-h| > a,$$

and ϕ and $\partial\phi/\partial x$ are continuous across the gaps. Here λ is the wavelength of the incident wave.

At $x = -\infty$ it is assumed that only a wave travelling away from the barriers exists. The situation is illustrated in figure 1.

It is convenient to split ϕ into symmetric and asymmetric terms. Thus we write

$$\phi(x,y) = \phi_s(x,y) + \phi_a(x,y),$$

where

$$\phi_s(x,y) = \phi_s(-x,y), \quad \phi_a(x,y) = -\phi_a(-x,y).$$

Then, for $x > b$, let

$$\phi_s(x,y) = \phi_0 e^{-Ky}\,(e^{-iK(x-b)} + R_s e^{iK(x-b)}) + \int_0^\infty \frac{s(k)\,e^{-k(x-b)}\,(k\cos ky - K\sin ky)}{k(k^2+K^2)}\,dk \tag{1}$$

and, for $|x| < b$, let

$$\phi_s(x,y) = \phi_0 S_1 e^{-Ky}\cos Kx + \int_0^\infty \frac{S(k)\cosh kx(k\cos ky - K\sin ky)}{k(k^2+K^2)}\,dk. \tag{2}$$

An integral equation for the function $U_s(y) \equiv (\partial\phi_s/\partial x)_{x=b}$ can be derived by applying the condition of continuity of U_s across $L = \{y|x = b, |y-h| < a\}$ by using the inversion formula first proved by Havelock (1929). If $U_s(y)$ is then normalized we obtain

$$\int_L u_s(t)\,G_s(y,t)\,dt = e^{-Ky}, \quad y \in L, \tag{3}$$

and

$$\int_L u_s(y)\,e^{-Ky}\,dy = C_s, \tag{4}$$

where C_s is real and is given by

$$C_s = -i(1-R_s)/\{(1+R_s) - i(1-R_s)\cot Kb\}, \tag{5}$$

$$G_s(y,t) = \frac{1}{\pi}\int_0^\infty \frac{(1+\coth kb)\,(k\cos kt - K\sin kt)\,(k\cos ky - K\sin ky)}{k(k^2+K^2)}\,dk$$

and

$$u_s(y) = 2iC_s U_s(y)/\phi_0(1-R_s).$$

The details are given in I, where expressions are given for the functions $s(k)$, S_1 and $S(k)$ in terms of $U_s(y)$.

Equations similar to (3) and (4) can be derived for the asymmetric potential ϕ_a, where all subscripts s are replaced by a, the kernel $G_a(y,t)$ is derived from $G_s(y,t)$ by replacing $\coth kb$ by $\tanh kb$, and C_a is derived from C_s by replacing $\cot Kb$ by $-\tan Kb$. Notice that as $x \to +\infty$

$$\phi(x,y) \sim 2\phi_0 e^{-Ky+iKb}(e^{-iKx} + R\,e^{iKx})$$

and as $x \to -\infty$

$$\phi(x,y) \sim 2\phi_0 T \exp(-Ky + iKb - iKx),$$

where

$$R = \tfrac{1}{2}(R_s + R_a)\,e^{-2iKb}, \quad T = \tfrac{1}{2}(R_s - R_a)\,e^{-2iKb}. \tag{6}$$

Up to this point, no approximation has been made and the exact linearized solution to the problem is embodied in (4) and (5), and the corresponding equations for the asymmetric case. These determine R_s and R_a once $u_s(y)$ and $u_a(y)$ have been obtained as the solution of the integral equation (3) and its counterpart in the asymmetric case. An explicit inversion of (3) is most unlikely so we assume at this point that $2a/h \ll 1$, whilst Kh and $2b/h$ are $O(1)$. In other words we are concerned with gaps small compared with the depth of immersion. This enables us to use an approximate method used by Packham & Williams (1972) to find C_s and C_a.

We can write

$$G_s(y,t) = -\pi^{-1}\log|y-t| + g_s(y,t),$$

where $g_s(y,t)$ is finite for $y, t \in L$. Then (4) becomes

$$\int_L u_s(t)\log|y-t|\,dt = -\pi e^{-Ky} + \pi\int_L u_s(t)\,g_s(y,t)\,dt, \quad y \in L.$$

For $2a/h \ll 1$, y and t differ little from h on L and we can replace $g_s(y,t)$ by $g_s(h,h)$, its value at the centre of the gap. In fact it can be shown that

$$g_s(y,t) = g_s(h,h) + O(1) \quad \text{as} \quad 2a/h \to 0, \quad y,t \in L.$$

Then $$\int_L u_s(t)\log|y-t|\,dt = \mu \equiv -\pi\left\{e^{-Kh} - \left(\int_L u_s(t)\,dt\right) g_s(h,h)\right\} \tag{7}$$

and a similar approximation applied to (4) gives

$$C_s = e^{-Kh}\int_L u_s(t)\,dt.$$

Now the integral equation (7) has the explicit solution (Cooke 1970)

$$u_s(y) = \mu\pi^{-1}[\log(\tfrac{1}{2}a)]^{-1}\{(y-h+a)(h+a-y)\}^{-\frac{1}{2}},$$

so that the solution satisfies

$$\int_L u_s(t)\,dt = \mu/\log(\tfrac{1}{2}a). \tag{8}$$

Elimination of μ between (7) and (8) gives for C_s the approximate form

$$C_s = e^{-2Kh}/\{g_s(h,h) - \pi^{-1}\log(\tfrac{1}{2}a)\}. \tag{9}$$

A similar result holds for C_a with $g_s(h,h)$ replaced by $g_a(h,h)$. We find that

$$g_{s,a}(h,h) = \frac{1}{\pi}\log 2h - \frac{2}{\pi}e^{-2Kh}\,\mathrm{Ei}\,(2Kh) + k_{s,a}(h,h). \tag{10}$$

where $$k_s(h,h) = \frac{1}{\pi}\int_0^\infty \frac{e^{-kb}(k\cos kh - K\sin kh)^2}{k\sinh kb\,(k^2+K^2)}\,dk$$

and $k_a(h,h)$ is derived from $k_s(h,h)$ by replacing $\sinh kb$ by $-\cosh kb$. Returning to (5) we see that $R_s = -(1-iA)/(1+iA)$ and $R_a = -(1+iB)/(1-iB)$, where $A = \cot Kb - C_s^{-1}$ and $B = \tan Kb + C_a^{-1}$. So from (6)

$$\left.\begin{aligned} |R| &= |1-AB|\,(1+A^2+B^2+A^2B^2)^{-\frac{1}{2}}, \\ |T| &= |A+B|\,(1+A^2+B^2+A^2B^2)^{-\frac{1}{2}}, \end{aligned}\right\} \tag{11}$$

so that $|R|^2 + |T^2| = 1$ as expected.

It follows from (11) that the condition for zero transmission is just $A+B = 0$, which gives

$$\begin{aligned} 2\operatorname{cosec} 2Kb &= C_s^{-1} - C_a^{-1} \\ &= \frac{2e^{2Kh}}{\pi}\int_0^\infty \frac{(k\cos kh - K\sin kh)^2}{k\sinh 2kb(k^2+K^2)}\,dk, \end{aligned} \tag{12}$$

where (9) and (10) have been used. It remains to show that (12) has solutions for certain values of the parameters Kh and $2b/h$. Notice that (12) is independent of the (small) gap size $2a/h$. It is convenient to introduce the dimensionless parameters $\alpha = 2Kb/\pi = 4b/\lambda$ and $\beta = \pi h/b$. The integral in (12) may be transformed into an infinite series using contour integration and we obtain as the condition for zero transmission

$$f(\alpha, \beta) \equiv \log\cosh(\tfrac{1}{2}\beta) - \alpha^{-1} + 2\sum_{n=1}^{\infty} \frac{(-1)^n \exp(-n\beta)}{n-\alpha} = 0. \tag{13}$$

A derivation of this result is given in the appendix.

Now from (13), as $\alpha \to 0, f(\alpha, \beta) \to -\infty$, and as $\alpha \to \infty, f(\alpha, \beta) \to \log\cosh\frac{1}{2}\beta$ unless α is close to an integer m. Thus if $\alpha = (2m-1) - \epsilon$ or $2m + \epsilon$, where ϵ is small and positive, then (13) shows that $f(\alpha, \beta) \to -\infty$ as $\epsilon \to 0$. Similarly if $\alpha = (2m-1)+\epsilon$ or $2m - \epsilon$, then $f(\alpha, \beta) \to \infty$ as $\epsilon \to 0$. These observations show that, for β sufficiently large, $f(\alpha, \beta) = 0$ for $\alpha = \alpha_m$ $(m = 1, 2, \ldots)$, and furthermore $\alpha_m \sim m - 1$ as $m \to \infty$. The roots disappear in pairs at discrete values of β as β decreases. For example, for $\beta = 2\pi$, $\alpha_1 = 0{\cdot}41$ and $\alpha_2 \simeq 1$ whereas, for $\beta = \pi$, α_1 and α_2 disappear and the first root is $\alpha_3 \simeq 2$.

Thus we see that for a given ratio of barrier spacing to gap depth (or given $\beta = \pi h/b$) there exists an infinity of values of the ratio of barrier separation to incident wavelength (or of $\alpha = 4b/\lambda$) for which total reflexion of the incident wave occurs.

The results of computing $|T|^2$, the proportion of wave energy transmitted through the gaps, from (11), are shown in figures 2 (*a*) and (*b*) for a gap width-to-depth ratio $2a/h$ of 0·05 and 0·15 respectively. Also shown are the corresponding curves due to Tuck (1971) for the single barrier ($2b/h = 0$). It is clear that, for a given gap size $2a/h$ and a given barrier spacing $2b/h$, the amount of wave energy transmitted is highly sensitive to changes in the wavenumber Kh of the incoming wave. The curves are seen to be dominated by the peaks of $|T|^2$ indicating total transmission of the incident wave corresponding to $AB = 1$ from (11).

The more unusual phenomenon of *zero* transmission, shown to exist by the argument following (13), does not appear on the curves as it first occurs at values of Kh for which $|T|^2$ is very small anyway. Because of this the significance of the phenomenon in the present problem is reduced somewhat. It ought to be possible to establish the existence of total reflexion when the gap size is not small, as was done in I, but only at the cost of losing the explicit simple result achieved here.

It can be seen that the effect of a second barrier with a gap, far from reducing the wave energy transmitted, can actually produce *total* transmission of the incident wave. It is possible, however, to reduce considerably the transmitted wave energy as can be seen from figure 2 (*a*). With a single barrier, and in the range $0{\cdot}2 < Kh < 0{\cdot}4$, never less than 36 % of the wave energy is transmitted. By adding a similar barrier with a spacing $2b/h = 6$ the transmitted wave energy is cut to less than 12 % in the same wavenumber range.

Conclusion

A simple approximate method has been used for calculating the surface wave energy transmitted through small gaps in two parallel vertical barriers. It has been shown that it is possible for an incident wave to be completely transmitted

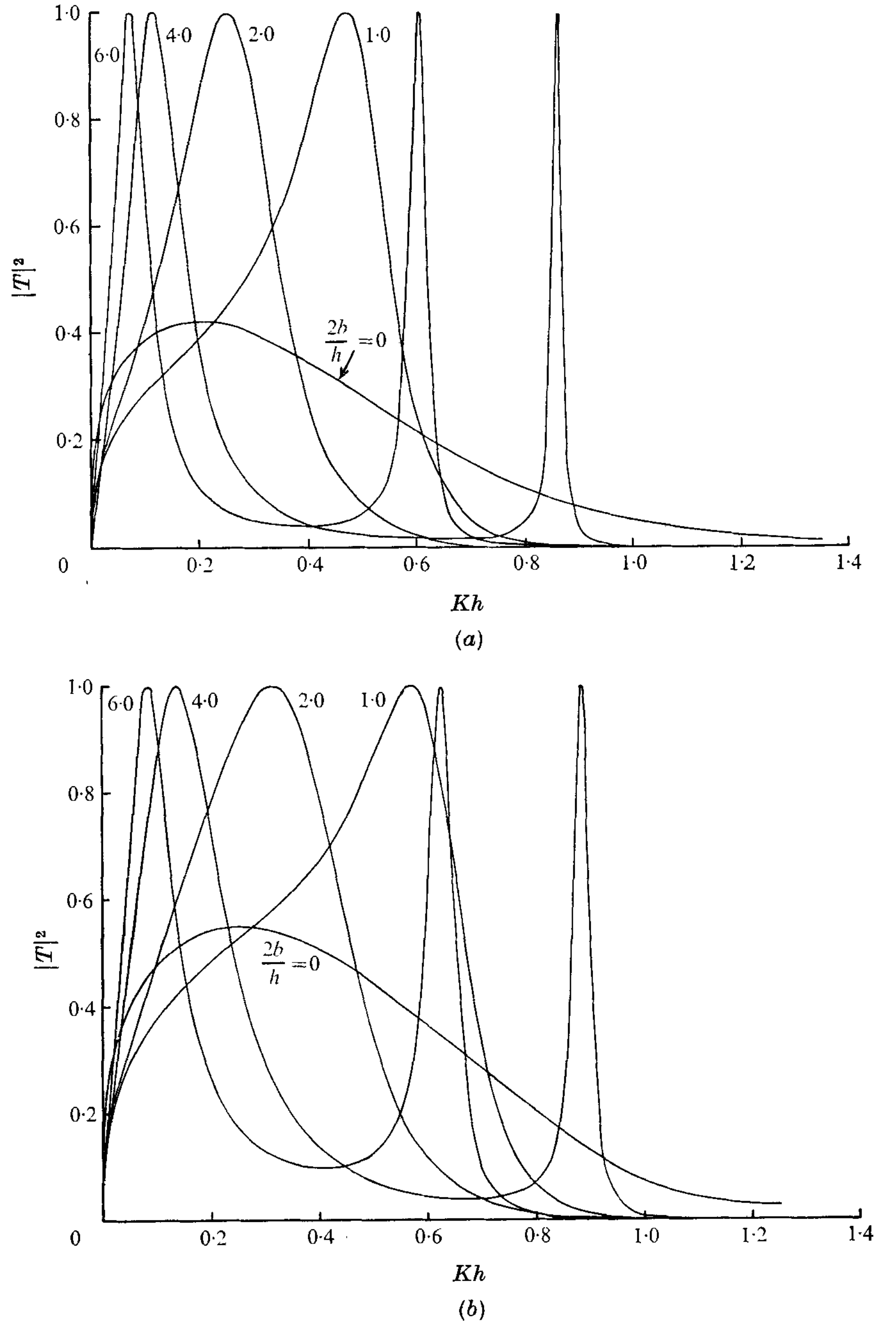

FIGURE 2. Proportion of wave energy transmitted. (a) $2a/h = 0{\cdot}05$. (b) $2a/h = 0{\cdot}15$.

through the small gaps. This result was not unexpected and occurs in similar problems in water-wave theory.

A more unusual result has also been shown, namely, that the incident wave can be totally reflected. For this to occur a certain transcendental equation had to be satisfied and it was shown that this equation was satisfied for an infinite number of configurations of barrier spacing, gap depth and incident wavelength. In the present problem, however, the solution being based on a small-gap approximation, total reflexion occurred at values of the parameters which were outside the range of physical interest.

Total reflexion has been proved to occur in the related problem of two finite parallel vertical barriers intersecting the free surface (Evans & Morris 1972). However, it turns out that the phenomenon does not occur in the problem studied by Jarvis (1971) of two infinite parallel vertical barriers extending downwards from points beneath the surface. This suggests that a possible criterion for total reflexion is two or more partial obstacles intersecting the free surface. As far as engineering applications are concerned, further studies on more realistic surface obstacles may well provide useful criteria for the more efficient design of breakwaters.

Appendix

The condition for zero transmission [equation (12)] may be written as

$$I = \int_0^\infty \frac{(k\cos kh - K\sin kh)^2}{k\sinh 2kb(k^2+K^2)}\,dk = \pi\exp(-2Kh)\operatorname{cosec}2Kb.$$

Now
$$I = I_0 + I_1,$$
where
$$I_0 = \int_0^\infty \frac{\sin^2 kh}{k\sinh 2kb}\,dk$$
and
$$I_1 = \int_0^\infty \frac{(k\cos 2kh - K\sin 2kh)}{\sinh 2kb(k^2+K^2)}\,dk.$$
We have
$$\frac{dI_0}{dh} = \int_0^\infty \frac{\sin 2kh}{\sinh 2kb}\,dk = \frac{\pi}{4b}\tanh\frac{\pi h}{2b}$$

(Gradshteyn & Ryzhik 1965, p. 503) and so $I_0 = \frac{1}{2}\log\cosh(\pi h/2b)$ since $I_0 = 0$ for $h = 0$. Also

$$I_1 = \lim_{\epsilon\to 0}\operatorname{Re}\int_\epsilon^\infty \frac{\exp(2ikh)\,dk}{(k-iK)\sinh 2kb}$$
$$= -\frac{\pi}{4Kb} + \frac{\pi\exp(-2Kh)}{\sin 2Kb} + \sum_{n=1}^{\infty}\frac{\pi(-1)^n\exp(-n\pi h/b)}{n\pi - 2Kb},$$

this result being obtained by rotating the path of integration through an angle $\frac{1}{2}\pi$. Substitution of the new parameters $\alpha = 2Kb/\pi$ and $\beta = \pi h/b$ now gives (13) as required.

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